Apr 14

§7.4 The maximum principle on noncompact manifolds

complete noncompact M and $\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0$
maximal principal fails $\longrightarrow$ sol. not unique

To obtain uniqueness of solution, we use the idea given by Li-Yau inequality

Next: introduce a growth rate control see (☆)
to get uniqueness

<u>Def</u> subsolution u: $Pu = \left(\frac{\partial}{\partial t} - \Delta\right) u \le 0$

Thm 7.39

If u is a smooth subsolution of the heat equation on $M^n \times [0,T]$ with $u(\cdot,0) \le 0$ and if

(☆) $$\int_0^T \int_{M^n} \exp(-\alpha d^2(x,O))\, u_+^2(x,t)\, d\mu(x)\, dt < \infty$$

for some $\alpha > 0$, then $u \le 0$ on $M^n \times [0,T]$

Rmk: in §3.6 and §7.1 we have seen the normalized flow converges exp fast.

Cor 7.40

fix some $O \in M$

- $Rc(x) \geq -C_1(1+d^2(x,O))$ for some C_1
- u bound subsolution
- IC $u(x,0) \leq 0$

$\Rightarrow$ $u(x,t) \leq 0$ bounded sol.'s are unique.

pf of Cor 7.40

Proof. Since $\operatorname{Rc}(x) \geq -C_1\left(1+r^2\right)$ on $B(O,r)$, a direct application of the volume comparison theorem implies that

Thm 1.132

$$\operatorname{Vol}(B(O,r)) \leq C_2 \exp\left(ar^2\right)$$

Can control the volume, then 7.25 is satisfied for some large alpha

for some $a=a(n,C_1)>0$ and C_2. It is then easy to see that the assumption of Theorem 7.39 holds for some α chosen suitably large. $\square$

Theorem 1.132 (Bishop volume comparison). *If (M^n,g) is a complete Riemannian manifold with $\operatorname{Rc} \geq (n-1)K$, where $K \in \mathbb{R}$, then for any $p \in M^n$, the volume ratio*

$$\frac{\operatorname{Vol}(B(p,r))}{\operatorname{Vol}_K(B(p_K,r))}$$

is a nonincreasing function of r, where p_K is a point in the n-dimensional simply connected space form of constant curvature K and Vol_K denotes the volume in the space form. In particular

$$\operatorname{Vol}(B(p,r)) \leq \operatorname{Vol}_K(B(p_K,r)) \tag{1.152}$$

for all $r>0$. Given p and $r>0$, equality holds in (1.152) if and only if $B(p,r)$ is isometric to $B(p_K,r)$.

uniqueness: if u_1, u_2 both solve

$$\begin{cases} Pu = 0 & \text{on } M \times [0,T) . \\ u(x,0) = u_0(x) \end{cases}$$

Take $w = u_1 - u_2$, then w satisfies

$$\text{(I)} \quad \begin{cases} Pw = 0 & \text{on } M \times [0,T] \\ w(x,0) = 0 \end{cases}$$

Apply maximum principle to (I) we get

$$w(x,t) \leq 0 \quad \text{on } M \times [0,T]$$

$$\Rightarrow \sup_{M \times [0,T]} w(x,t) \leq 0$$

Do the same to $-w$, we get

$$\sup_{M \times [0,T]} -w(x,t) \leq 0$$

This means $w = 0$ on $M \times [0,T)$.

pf. of Thm 7.39

- $h = -\dfrac{d^2(x,0)}{4(2\tau - t)}$ locally Lipschitz on $M^n \times [0, 2\tau)$

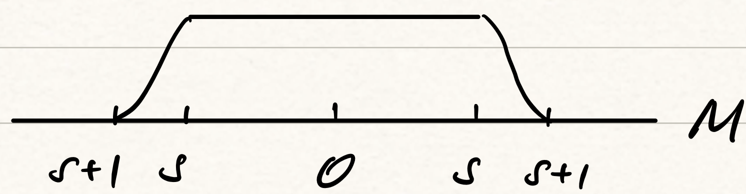
  $\tau > 0$, small (later)

  $|\nabla d(\,,0)| = 1 \Rightarrow |\nabla h|^2 + \dfrac{\partial h}{\partial t} = 0 \quad (\star\star)$

  x

  $\dfrac{d}{ds}\Big|_{s=0} d(\gamma(s), 0) = \langle \nabla d(\gamma(0), 0), \gamma'(0) \rangle$

- cut-off φ_s, $|\nabla \varphi_s| \le 2$



- Consider $\dfrac{\partial u}{\partial t} - \Delta u \le 0$

  $$\Rightarrow \int_0^\tau \int_M \left(\frac{\partial u}{\partial t} - \Delta u \right) \cdot \varphi_s e^h \le 0$$

IBP + Cauchy-Schwartz $\Rightarrow$

$$0 \ge \int_0^\tau \int_M e^h \left(-|2\nabla \varphi_s|^2 u_+^2 - \frac{1}{2} \varphi_s^2 u_+^2 |\nabla h|^2 \right) d\mu \, dt$$
$$- \frac{1}{2} \int_0^\tau \int_M \varphi_s^2 e^h u_+^2 \frac{\partial h}{\partial t} \, d\mu \, dt$$

$(\star\star)$ + $u_+(0) = 0 \Rightarrow$

$$\left(\int_M \varphi_s^2 e^h u_+^2 \, d\mu \right)(\tau) \le 4 \int_0^\tau \int_M e^h u_+^2 |\nabla \varphi_s|^2 \, d\mu \, dt$$

a function of τ

$$h = -\frac{d^2(x,0)}{4(2\tau - t)} \le -\frac{d^2(x,0)}{8\tau}$$

for $\tau \le \frac{1}{8\alpha}$, $e^h \le e^{-d^2(x,0)/8\tau} \le e^{-\alpha}$

we have

$$\left(\int_{M^n} \varphi_s^2 e^h u_+^2\right)(\tau) \le 16 \int_0^\tau \underbrace{\int_{B(0,s+1)\setminus B(0,s)}}_{0 \le \varphi_s < 1} e^{-\alpha d^2(x,0)} u_+^2 \, d\mu \, dt$$

$$\xrightarrow{s \to \infty} 0$$

LHS ≤ 0 as $s \to \infty$. (supp $\varphi_s = B(0,s) \to M$)

$\Rightarrow$ $u_+ \equiv 0$ on $M^n \times [0,\tau]$

$\Rightarrow$ $u \le 0$ for $t \in [0, \min\{\tau, T\}]$.

if $\tau < T$ take τ to be the initial time

\+ induction.

$\Rightarrow$ $u \le 0$ on $(0,T)$.

Remark 7.41. *In* **[375]**, *Li and Yau proved the uniqueness of solutions which are bounded from below under a certain lower bound assumption on the Ricci curvature. The key idea is that one can obtain growth control of positive solutions to the heat equation by their gradient estimates (also called Li-Yau inequalities).*

Ric bounded from below $\Rightarrow$ gradient estimate

$\Rightarrow$ growth control of $u_+ = \max\{0, u\}$

solution of heat eqn

Maximum principle

maximum value of a subsolution of the heat equation cannot increase over time

General maximum principle for heat eqn with time-dependent Laplacian

$(M^n, g(t))$ $\quad t \in [0, T)$. $\quad$ smooth

$$R_*(t) = \inf_{x \in M} (g^{ij} \Upsilon_{ij})(x, t)$$

$$\frac{\partial}{\partial t} g_{ij} = -2 \Upsilon_{ij}$$

Thm 7.42.

- For $t \in [0, T)$, $g(t) \geq g^*$ complete on M
- R_* finite and integrable on $[0, T)$.
- $u(x, t)$ Lipschitz weak solution to

$$\frac{\partial}{\partial t} u \leq \Delta_{g(t)} u.$$

- If $u(\cdot, 0) \leq 0$, and $\exists\, \alpha > 0$ s.t. for some fixed $\mathcal{O} \in M$

$$(*) \quad \int_0^T \int_{M^n} \exp(-\alpha d_*^2(x, \mathcal{O}))\, u_+^2(x, t)\, d\mu(x)\, dt < \infty$$

distance w.r.t. g^*.

▷ then $u(x, t) \leq 0$ on $M \times [0, T]$